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Journal of Approximation Theory 132 (2005) 72 – 76
Journal of Approximation Theory
www.elsevier.com/locate/jat

Smooth planar r-splines of degree $2r$

Ştefan O. Tohăneanu

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

Received 24 February 2004; received in revised form 4 October 2004; accepted in revised form 27 October 2004

Communicated by Amos Ron

Abstract

Alfeld and Schumaker [Numer. Math. 57 (1990) 651–661] give a for mula for the dimension of the space of piecewise polynomial functions (splines) of degree d and smoothness r on a generic triangulation of a planar simplicial complex Δ (for $d \geqslant 3r+1$) and any triangulation (for $d \geqslant 3r+2$). In Schenck and Stiller [Manuscripta Math. 107 (2002) 43–58], it was conjectured that the Alfeld–Schumaker formula actually holds for all $d \geqslant 2r+1$. In this note, we show that this is the best result possible; in particular, there exists a simplicial complex Δ such that for any r, the dimension of the spline space in degree $d = 2r$ is not given by the formula of Alfeld and Schumaker [Numer. Math. 57 (1990) 651–661]. The proof relies on the explicit computation of the nonvanishing of the first local cohomology module described in Schenck and Stillman [J. Pure Appl. Algebra 117 & 118 (1997) 535–548].
Published by Elsevier Inc.

MSC: primary 13D40; secondary 52B20

Keywords: Simplicial complex; Bivariate spline; Hilbert function

1. Introduction and preliminaries

Let Δ be a planar, strongly connected finite-simplicial complex. The set of piecewise polynomial functions on Δ of smoothness r has the structure of a module $C^r(\Delta)$ over the polynomial ring $\mathbb{R}[x, y]$; the subset of $f \in C^r(\Delta)$ such that $f|_\sigma$ is of degree at most d for each two-simplex $\sigma \in \Delta$ is a finite-dimensional real vector-space, denoted $C^r(\Delta)_d$. In [2], for almost all triangulations, Alfeld and Schumaker give a formula for the dimension of the $C^r(\Delta)_d$ in terms of combinatorial and local geometric data (data depending only on local

E-mail address: tohanean@math.tamu.edu.

0021-9045/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.jat.2004.10.011

geometry at the interior vertices of Δ), as long as $d \geqslant 3r+1$. In [12], it was conjectured that their formula actually holds as long as $d \geqslant 2r+1$. The purpose of this brief note is to show that this conjecture is optimal; in particular, we exhibit a planar simplicial complex Δ such that for *any* r there exist "special" splines in degree $d = 2r$. In other words, the conjecture of Schenck and Stiller [12] is tight.

The methods we use depend on the homological approach developed by Billera in [3] to answer a conjecture of Strang. This approach was further developed by Schenck and Stillman in the papers [13,14], using a chain complex different from Billera's and some additional technical tools (local cohomology and duality). When Δ is a planar simplicial complex, it turns out that the delicate geometry of the problem is captured by a certain local cohomology module, which as shown in [14] has a simple description (see below). For example, the geometry of the famous "Morgan–Scott" example, which shows that even for $r = 1$ the formula of Alfeld and Schumaker [2] does not apply for $d = 2$, is captured by this local cohomology. For more results on the Morgan–Scott triangulation see [7,8], and for computations of the dimension for small triangulations see [1]. In the next section we quickly review the presentation of this module; then we exhibit a specific Δ and prove that for any r, the dimension of this module (which exactly captures the discrepancy between Alfeld–Schumaker's formula and the actual dimension) is nonzero in degree $d = 2r$.

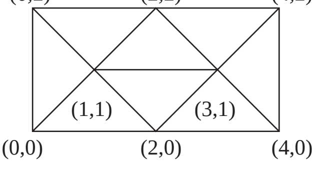
2. Review of local cohomology

By taking the cone $\hat{\Delta}$ over Δ, we turn the problem of computing dim $C^r(\Delta)_d$ into a problem in commutative algebra—compute the Hilbert function $C^r(\hat{\Delta})_d$ (for background on this, See [11]). As shown in [14],

$$\dim\, C^r(\Delta)_d = \dim\, C^r(\hat{\Delta})_d = L(\Delta, r, d) + \dim\, N_d,$$

where $L(\Delta, r, d)$ is the Alfeld–Schumaker formula and N is a graded $R = \mathbb{R}[x, y, z]$ module of finite length. Lemma 3.8 of Schenck and Stillman [14] contains the following description: N is the quotient of a free module generated by the totally interior edges (those edges with no vertex $\subseteq \partial\Delta$), modulo the syzygies at each interior vertex. The generators of N are shifted so that they have degree $r+1$. This description seems cumbersome, but as we will see in the example below, it is fairly easy to work with.

So in the terms above, the conjecture of Schenck and Stiller [12] is that N vanishes in degree $2r+1$. Our goal is to show that this bound is the best possible, so we want to find a configuration Δ such that for all r, $N_{2r} \neq 0$. Consider the following simplicial complex (see also [12]).



To find N, we begin by determining the minimal free resolutions for the ideals $I_i = \mathcal{J}(v_i)$ for v_1 and v_2 the interior vertices. We have

$$I_1 = \langle (x+y-2z)^{r+1}, (x-y)^{r+1}, (y-z)^{r+1} \rangle,$$

$$I_2 = \langle (x+y-4z)^{r+1}, (x-y-2z)^{r+1}, (y-z)^{r+1} \rangle.$$

These ideals are in $R = \mathbb{R}[x, y, z]$. Notice that $y - z$ is the linear form vanishing on the totally interior edge. With the change of variables given by the matrix

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 2 \\ 1 & 1 & -4 \end{bmatrix},$$

we can suppose that

$$I_1 = \langle x^{r+1}, (x+y)^{r+1}, y^{r+1} \rangle \quad \text{and} \quad I_2 = \langle z^{r+1}, (z+y)^{r+1}, y^{r+1} \rangle.$$

The minimal free resolutions for these ideals are:

$$0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^{r+1} & (x+y)^{r+1} & y^{r+1} \end{bmatrix}} R \longrightarrow R/I_1 \longrightarrow 0$$

and

$$0 \longrightarrow R^2 \xrightarrow{\begin{bmatrix} A_2 & D_2 \\ B_2 & E_2 \\ C_2 & F_2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} z^{r+1} & (z+y)^{r+1} & y^{r+1} \end{bmatrix}} R \longrightarrow R/I_2 \longrightarrow 0.$$

By Schenck and Stillman [14], Lemma 3.8, $N \approx R(-r-1)/\langle C_1, F_1, C_2, F_2 \rangle$. In what follows we will prove that the Hilbert function of $R/\langle C_1, F_1, C_2, F_2 \rangle$ is nonzero in degree $r-1$, for any positive integer r. In other words

$$\begin{aligned} HF(N, 2r) &= HF(R(-r-1)/\langle C_1, F_1, C_2, F_2 \rangle, 2r) \\ &= HF(R/\langle C_1, F_1, C_2, F_2 \rangle, r-1) \\ &\neq 0. \end{aligned}$$

3. The Hilbert function of $R/\langle C_1, F_1, C_2, F_2 \rangle$ is nonzero in degree $r-1$

In the previous section, we saw that the ideals I_1 and I_2 have a special form. First, notice that they are symmetric (in terms of the generators) in x and z. So replacing x by z in the forms of C_1 and F_1 we obtain C_2 and F_2. Next, observe that we can look at the ideal I_1 as an ideal in $A = \mathbb{R}[x, y]$. Similarly, I_2 is an ideal $A' = \mathbb{R}[y, z]$. Hence $C_1, F_1 \in A$ and $C_2, F_2 \in A'$.

For $i = 1, 2$, the ideal $\langle C_i, F_i\rangle$ is a complete intersection. For example, if $\langle C_1, F_1\rangle$ is not a complete intersection, since $C_1 \neq 0$ and $F_1 \neq 0$, then $codim(\langle C_1, F_1\rangle) = 1$. Therefore, there is a nonunit in A, say d_1, such that $d_1|C_1$ and $d_1|F_1$. Therefore, $d_1|x^{r+1} = B_1F_1 - E_1C_1$ and $d_1|(x+y)^{r+1} = A_1F_1 - D_1C_1$. Hence $codim(\langle x^{r+1}, (x+y)^{r+1}\rangle) = 1$. But this contradicts the fact that $codim(\langle x^{r+1}, (x+y)^{r+1}\rangle) = 2$, as $\{x^{r+1}, (x+y)^{r+1}\}$ is a regular A-sequence. So the ideal $\langle C_1, F_1\rangle$ is a complete intersection. The same argument shows that $\langle C_2, F_2\rangle$ is also a complete intersection.

These observations will simplify our future computations. We will need to discuss two cases, depending on if r is odd or even.

3.1. $r+1 = 2n$. Let $A = \mathbb{R}[x, y]$ and $I_1 = \langle x^{2n}, (x+y)^{2n}, y^{2n}\rangle$. By Schenck and Stillman [13], Theorem 3.1, a free resolution for I_1 is:

$$0 \longrightarrow A(-3n)^2 \xrightarrow{\begin{bmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{bmatrix}} A(-2n)^3 \xrightarrow{\begin{bmatrix} x^{2n} & (x+y)^{2n} & y^{2n} \end{bmatrix}} I_1 \longrightarrow 0,$$

where $\deg C_1 = \deg F_1 = 3n - 2n = n$. From the observations at the beginning of this section we get the minimal free resolution for $A/\langle C_1, F_1\rangle$:

$$0 \longrightarrow A(-2n) \longrightarrow A(-n)^2 \longrightarrow A \longrightarrow A/\langle C_1, F_1\rangle.$$

Therefore, the Hilbert series is $HS(A/\langle C_1, F_1\rangle, t) = \frac{1-2t^n+t^{2n}}{(1-t)^2}$. Hence, there exists a monomial $x^u y^v$ of degree $2n-2 = r-1$ which is not in $\langle C_1, F_1\rangle$. In fact, with an easy computation we can see that this monomial is actually x^{2n-2}.

All we did here is in two variables x and y. Let us go back to the ring $R = \mathbb{R}[x, y, z]$ and suppose that the above monomial $x^u y^v$ is in $\langle C_1, F_1\rangle + \langle C_2, F_2\rangle = \langle C_1, F_1, C_2, F_2\rangle$. Then there are $\alpha_1, \beta_1, \alpha_2, \beta_2 \in R$ such that

$$x^u y^v = \alpha_1 C_1 + \beta_1 F_1 + \alpha_2 C_2 + \beta_2 F_2.$$

In this equation, since for $x = z$ we get $C_1 = C_2$ and $F_1 = F_2$ (see the remarks at the beginning), we obtain an equation in $A = \mathbb{R}[x, y]$:

$$\begin{aligned} x^u y^v &= \alpha_1(x, y, x)C_1 + \beta_1(x, y, x)F_1 + \alpha_2(x, y, x)C_1 + \beta_2(x, y, x)F_1. \\ &= \alpha_1' C_1 + \beta_1' F_1. \end{aligned}$$

So $x^u y^v \in \langle C_1, F_1\rangle$. This contradicts the way we chose $x^u y^v$. Hence, there is a monomial of degree $r-1$ which is not in $\langle C_1, F_1, C_2, F_2\rangle$.

3.2. $r+1 = 2n+1$. For the odd case, the idea is almost identical. Let $A = \mathbb{R}[x, y]$ and $I_1 = \langle x^{2n+1}, (x+y)^{2n+1}, y^{2n+1}\rangle$. Again, by Schenck and Stillman [13], Theorem 3.1, a free resolution for A/I_1 is:

$$0 \longrightarrow \begin{matrix} A(-3n-1) \\ \oplus \\ A(-3n-2) \end{matrix} \xrightarrow{\begin{bmatrix} A_1 & D_1 \\ B_1 & E_1 \\ C_1 & F_1 \end{bmatrix}} A(-2n-1)^3 \xrightarrow{[I_1]} A \longrightarrow A/I_1 \longrightarrow 0,$$

where $\deg C_1 = 3n + 1 - (2n + 1) = n$ and $\deg F_1 = 3n + 2 - (2n + 1) = n + 1$. $\langle C_1, F_1\rangle$ is a complete intersection so the minimal free resolution for $A/\langle C_1, F_1\rangle$ is:

$$0 \longrightarrow A(-2n-1) \longrightarrow A(-n) \oplus A(-n-1) \longrightarrow A \longrightarrow A/\langle C_1, F_1\rangle.$$

Therefore, the Hilbert series is $HS(A/\langle C_1, F_1\rangle, t) = \frac{1-t^n-t^{n+1}+t^{2n+1}}{(1-t)^2}$. Hence, there exists a monomial $x^u y^v$ of degree $2n - 1 = r - 1$ which is not in $\langle C_1, F_1\rangle$. As in 3.1., the same argument gives us that in fact this monomial is not in $\langle C_1, F_1, C_2, F_2\rangle$.

In conclusion the Hilbert function of $R/\langle C_1, F_1, C_2, F_2\rangle$ is nonzero in degree $r - 1$. This is exactly what we wanted to see.

Acknowledgments

I thank Hal Schenck for helpful comments and discussions and for introduction to the subject. This research was partially supported by NSF Grant DMS-03-11142 and ATP Grant 010366-0103. I thank two anonymous referees for useful suggestions.

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